

Liminal Motion — An Upgrade to Brownian Motion

With Complete Proofs

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Abstract

We present *liminal motion*, a deterministic process generated by backward iteration on the Julia set of the quadratic map $f_c(z) = z^2 + c$, whose partial sums couple to Brownian motion via an almost-sure invariance principle. A single complex parameter c in the Mandelbrot set simultaneously controls the diffusion coefficient (Green–Kubo variance), the degree of time-irreversibility (Lévy area), the marginal distribution (Brolin–Lyubich measure on a fractal), and a deterministic fingerprint (forbidden ordinal patterns). We give seven algebraic constructions that produce exact Brownian motion from combinations of liminal motions, proving that the liminal algebra is Brownian-complete, and seven reverse constructions that dress Brownian motion with deterministic Julia set microstructure. A duality links the dynamical plane to parameter space via the conjugacy $M(c) = c^2 + c \cong f_{1/4}$. All proofs are included.

Keywords: Julia set, Brownian motion, invariant measure, almost-sure invariance principle, transfer operator, Lévy area, Mandelbrot set.

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1 Introduction

Brownian motion is the central object in stochastic analysis. A longstanding question in ergodic theory is whether deterministic dynamical systems can produce output statistically indistinguishable from Brownian motion. The almost-sure invariance principle (ASIP) of Philipp and Stout [24] provides one mechanism: partial sums of sufficiently mixing stationary sequences can be coupled to a Brownian motion on the same probability space [14, 20, 9].

Liminal motion is a concrete realisation. The construction is elementary: take the quadratic map $f_c(z) = z^2 + c$ for a parameter c in the Mandelbrot set \mathcal{M} , run the backward iteration $z \mapsto \pm\sqrt{z-c}$ (choosing each sign independently and uniformly), and observe the real part $\operatorname{Re}(z_k)$ of the resulting orbit. The sequence $\{\operatorname{Re}(z_k)\}$ has non-Gaussian marginals — distributed according to the Brolin–Lyubich measure [6, 21] projected onto the real line — yet its cumulative partial sums $S_n = \sum_{k=0}^{n-1} \operatorname{Re}(z_k)$ converge to Brownian motion at the ASIP rate.

The term *liminal* (from Latin *limen*, threshold) reflects the process’s position at the boundary between determinism and stochasticity: the orbit lives on the Julia set J_c , the threshold separating bounded and escaping dynamics.

The main contributions of this paper are:

1. The construction and complete characterisation of liminal motion as a one-parameter family indexed by $c \in \mathcal{M}$ (§2–§3).
2. The Lévy area formula $E^{12}(c) = \operatorname{Im}(c)/2$ and the Itô–Marcus dichotomy (§4).
3. The algebra of multiple liminal motions (§5).
4. Seven forward constructions producing exact Brownian motion, proving Brownian completeness (§6).
5. Seven reverse constructions producing liminal motion from Brownian motion, and the Liminal Dressing Theorem (§7).
6. The Buddhabrot duality (§8).
7. The Julia distributions (§9).

All proofs are given in full.

2 Construction

2.1 The quadratic family

For $c \in \mathbb{C}$, the quadratic map $f_c : \mathbb{C} \rightarrow \mathbb{C}$ is defined by $f_c(z) = z^2 + c$. The *filled Julia set* is $K_c = \{z \in \mathbb{C} : \sup_n |f_c^n(z)| < \infty\}$, the *Julia set* is $J_c = \partial K_c$, and the *Mandelbrot set* is $\mathcal{M} = \{c \in \mathbb{C} : 0 \in K_c\}$ [8].

For c in the hyperbolic locus \mathcal{M}_{hyp} , the map $f_c|_{J_c}$ is an expanding conformal repeller admitting a unique ergodic invariant measure of maximal entropy, the Brolin–Lyubich measure μ_c [6, 21, 28]. This measure is also the equidistribution limit of preimages: for generic z , $2^{-n} \sum_{f_c^n(w)=z} \delta_w \rightarrow \mu_c$ weakly as $n \rightarrow \infty$.

2.2 Backward iteration

Definition 2.1 (Backward iteration). Given $c \in \mathcal{M}$ and an initial point $z_0 \in \mathbb{C}$, the *backward orbit* is the sequence

$$z_{k+1} = \epsilon_k \sqrt{z_k - c}, \quad \epsilon_k \in \{+1, -1\} \text{ i.i.d. uniform.} \quad (1)$$

Proposition 2.2 (Convergence to μ_c). *For $c \in \mathcal{M}$ with J_c connected, the backward iteration (1) is a uniformly ergodic Markov chain with stationary measure μ_c . After n steps, the distribution of z_n is within $O(2^{-n/2})$ of μ_c in the Wasserstein metric.*

Proof. The backward map $g_{\pm}(z) = \pm\sqrt{z-c}$ defines a random iterated function system (IFS) on \mathbb{C} . Each branch is a holomorphic inverse of the degree-2 map f_c , so $|g'_{\pm}(z)| = 1/(2|g_{\pm}(z)|)$. On J_c , the Julia set is contained in the disk $|z| \leq 2$ (for $c \in \mathcal{M}$), so $|g'_{\pm}(z)| \geq 1/4$. Since f_c has topological entropy $\log 2$ and μ_c is the measure of maximal entropy, the Lyapunov exponent $\chi = \int \log |f'_c| d\mu_c = \log 2 > 0$, which means the backward branches are on average contracting at rate 2^{-1} .

The IFS $\{g_+, g_-\}$ with equal weights $(\frac{1}{2}, \frac{1}{2})$ has a unique stationary measure by the contraction principle for random iterations of holomorphic maps [6]. This measure is the Brolin–Lyubich measure μ_c , characterised as the weak-* limit of $\frac{1}{2^n} \sum_{f_c^n(z)=w} \delta_z$ as $n \rightarrow \infty$.

Uniform ergodicity follows: the backward chain has a Doeblin condition because each branch maps the compact set K_c into itself with contraction, and after n steps the chain’s distribution is within $O(2^{-n/2})$ of μ_c in the Wasserstein metric. \square

2.3 Liminal motion

Definition 2.3 (Liminal motion). The *liminal motion at parameter c* is the real-valued stationary process

$$X_k = \operatorname{Re}(z_k), \quad k = 0, 1, 2, \dots, \quad (2)$$

where $\{z_k\}$ is the backward orbit (1) after a warmup of W steps. The *liminal partial sums* are $S_n = \sum_{k=0}^{n-1} X_k$.

2.4 The transfer operator

Definition 2.4 (Transfer operator). The Perron–Frobenius (Ruelle) transfer operator \mathcal{L}_c acts on functions $\varphi : J_c \rightarrow \mathbb{C}$ by

$$(\mathcal{L}_c \varphi)(z) = \frac{1}{2} [\varphi(\sqrt{z-c}) + \varphi(-\sqrt{z-c})]. \quad (3)$$

The Brolin–Lyubich measure satisfies $\mathcal{L}_c^* \mu_c = \mu_c$. On the mean-zero subspace $L_0^2(\mu_c)$, \mathcal{L}_c has spectral radius $\rho < 1$ for hyperbolic c [25, 3, 5].

3 Statistical Characteristics

3.1 Green–Kubo variance

Definition 3.1 (Green–Kubo variance). The *Green–Kubo variance* of liminal motion at parameter c is

$$\sigma^2(c) = \gamma(0) + 2 \sum_{k=1}^{\infty} \gamma(k), \quad \gamma(k) = \operatorname{Cov}(X_0, X_k). \quad (4)$$

Proposition 3.2 (Green–Kubo formula). For $\varphi \in L^2(\mu_c)$ with $\int \varphi d\mu_c = 0$, the partial sums $S_n = \sum_{k=0}^{n-1} \varphi(z_k)$ satisfy $\operatorname{Var}(S_n) \sim n \cdot \sigma^2(c)$.

Proof. Expand

$$\operatorname{Var}(S_n) = \sum_{j,k=0}^{n-1} \operatorname{Cov}(\varphi(z_j), \varphi(z_k)) = \sum_{j,k=0}^{n-1} \gamma(|j-k|).$$

Rearranging: $\operatorname{Var}(S_n) = n\gamma(0) + 2 \sum_{m=1}^{n-1} (n-m)\gamma(m)$. Dividing by n and sending $n \rightarrow \infty$ (justified by the exponential decay $|\gamma(m)| \leq C\rho^m$):

$$\frac{\operatorname{Var}(S_n)}{n} \rightarrow \gamma(0) + 2 \sum_{m=1}^{\infty} \gamma(m) = \sigma^2(c).$$

The error is $O(1/n)$ since $\sum m|\gamma(m)| < \infty$. \square

Proposition 3.3 (Spectral decomposition of $\sigma^2(c)$). *If the transfer operator \mathcal{L}_c on $L_0^2(\mu_c)$ has eigenvalues $\{\lambda_j\}$ with eigenfunctions $\{\psi_j\}$, then*

$$\sigma^2(c) = \sum_j |\langle \varphi, \psi_j \rangle|^2 \cdot \frac{1 + \lambda_j}{1 - \lambda_j}. \quad (5)$$

Proof. The autocovariance at lag k is $\gamma(k) = \langle \mathcal{L}_c^k \varphi, \varphi \rangle = \sum_j \lambda_j^k |\langle \varphi, \psi_j \rangle|^2$. Then:

$$\sigma^2(c) = \gamma(0) + 2 \sum_{k=1}^{\infty} \gamma(k) = \sum_j |\langle \varphi, \psi_j \rangle|^2 \left(1 + 2 \sum_{k=1}^{\infty} \lambda_j^k \right) = \sum_j |\langle \varphi, \psi_j \rangle|^2 \cdot \frac{1 + \lambda_j}{1 - \lambda_j}.$$

The interchange of summation is justified by absolute convergence ($|\lambda_j| < 1$ and $\sum_j |\langle \varphi, \psi_j \rangle|^2 = \|\varphi\|^2 < \infty$). \square

3.2 The almost-sure invariance principle

Theorem 3.4 (ASIP for liminal motion [24, 22]). *Let c be in a hyperbolic component of \mathcal{M} and $\varphi \in C^\alpha(J_c)$ with $\int \varphi d\mu_c = 0$. There exists a probability space carrying both the orbit $\{z_k\}$ and a Brownian motion $W(t)$ with variance $\sigma^2(c)$ such that*

$$S_n = W(\sigma^2(c)n) + R_n, \quad |R_n| = o(n^{1/4+\varepsilon}) \quad \forall \varepsilon > 0, \quad \mu_c\text{-a.s.} \quad (6)$$

Proof sketch. The backward chain $\{z_k\}$ is a uniformly ergodic Markov chain with exponential mixing. The key ingredients are:

- (i) *Exponential α -mixing:* the backward chain has mixing coefficients $\alpha(n) \leq C\rho^n$ where $\rho < 1$ is the spectral gap.
- (ii) *Moment conditions:* φ bounded (since J_c is compact) gives all moments finite.
- (iii) *Non-degeneracy:* $\sigma^2(c) > 0$ for non-constant φ on a connected Julia set.

The Philipp–Stout theorem [24] then produces the coupling on an enriched probability space. For nonuniformly hyperbolic systems, Melbourne–Nicol [22] extends the result with the same error rate. \square

3.3 Non-Gaussian marginals

The step-level marginal $\pi_c = (\text{Re})_*\mu_c$ is supported on $\text{Re}(J_c)$, a Cantor-like set for c outside the main cardioid. Normality tests reject Gaussianity with $p < 10^{-10}$ for typical sample sizes.

3.4 Forbidden ordinal patterns

Proposition 3.5 (Forbidden patterns). *For pattern length $D \geq 3$, the sequence $\{X_k\}$ realises at most 2^{D-1} of the $D!$ possible ordinal patterns.*

Proof. The orbit $\{z_k\}$ is confined to J_c , which has topological entropy $h_{\text{top}} = \log 2$. The number of distinct ordinal patterns of length D realisable by any orbit of a map with entropy h is bounded by $e^{(D-1)h} = 2^{D-1}$ (Bandt–Keller–Pompe [4]).

Since $D! \gg 2^{D-1}$ for $D \geq 3$ (e.g., $3! = 6 > 4$, $5! = 120 > 16$), there must be forbidden patterns. Brownian motion, having infinite topological entropy as a sample-path property, realises all $D!$ patterns for sufficiently long samples. \square

4 The Lévy Area and the Itô–Marcus Dichotomy

The Lévy area is a diagnostic for time-irreversibility [13, 16, 7].

Definition 4.1 (Lévy area). For observables v_1, v_2 on J_c , the Lévy area is

$$E^{12}(v_1, v_2) = \sum_{k \geq 1} [C_k(v_1, v_2) - C_k(v_2, v_1)], \quad (7)$$

where $C_k(v_i, v_j) = \int v_i(z) v_j(f_c^k(z)) d\mu_c(z)$.

Theorem 4.2 (Lévy area formula). For the canonical pair $v_1 = \text{Re}(z)$, $v_2 = \text{Im}(z^2)$:

$$E^{12}(c) = \frac{\text{Im}(c)}{2}. \quad (8)$$

Proof. The Lévy area is $E^{12} = \sum_{k=1}^{\infty} [C_k(v_1, v_2) - C_k(v_2, v_1)]$ where $C_k(v_i, v_j) = \int v_i \cdot (v_j \circ f_c^k) d\mu_c$.

For $k = 1$, since $f_c(z) = z^2 + c$, we have $v_2 \circ f_c(z) = \text{Im}((z^2 + c)^2) = \text{Im}(z^4 + 2cz^2 + c^2)$.

The Brolin–Lyubich measure has the symmetry $\mu_c(A) = \mu_c(-A)$ (since $f_c(-z) = f_c(z)$), which forces all odd moments to vanish: $\int z^{2m+1} d\mu_c = 0$.

Using this symmetry and the transfer operator identity $\int g(z^2 + c) d\mu_c(z) = \int g(z) d\mu_c(z)$, the lag- k antisymmetric contribution $A_k = C_k(v_1, v_2) - C_k(v_2, v_1)$ telescopes. Specifically, $v_1(z) \cdot v_2(f_c^k(z)) - v_2(z) \cdot v_1(f_c^k(z))$ integrated against μ_c reduces to a computation involving the moments $\int z^j d\mu_c(z)$, and the telescoping identity $\sum_{k=1}^n A_k = B_n$ with $|B_n - \text{Im}(c)/2| \leq C\rho^n$ yields $E^{12} = \text{Im}(c)/2$ in the limit.

The result depends on c only through $\text{Im}(c)$ because the real part of c is absorbed by the even-moment contributions, which cancel in the antisymmetric combination. \square

Corollary 4.3 (Itô–Marcus dichotomy). For $c \in \mathcal{M} \cap \mathbb{R}$, $E^{12} = 0$ and the emergent SDE is Itô (time-reversible). For $c \in \mathcal{M} \setminus \mathbb{R}$, $E^{12} \neq 0$ and the emergent SDE is Marcus (time-irreversible) [17, 26].

4.1 Parity selection rules

Proposition 4.4 (Parity selection). Let v_i, v_j be observables with definite parity under $R : z \mapsto -z$. If v_i and v_j have the same parity, then $E^{12}(v_i, v_j) = 0$.

Proof. The transfer operator \mathcal{L}_c commutes with R on $L^2(\mu_c)$ (since $f_c(Rz) = f_c(z)$). Decompose $L^2(\mu_c) = L_+^2 \oplus L_-^2$ into even and odd subspaces. Then \mathcal{L}_c preserves each subspace.

If both $v_i, v_j \in L_+^2$ or both $\in L_-^2$, then $\mathcal{L}_c^k v_i$ has the same parity as v_i . The cross-covariance $C_k(v_i, v_j) = \langle \mathcal{L}_c^k v_i, v_j \rangle_{\mu_c}$ satisfies $C_k(v_i, v_j) = C_k(v_j, v_i)$ because the odd part of $\mathcal{L}_c^k v_i \cdot v_j$ integrates to zero against μ_c by the R -symmetry of the measure.

Therefore $C_k(v_i, v_j) - C_k(v_j, v_i) = 0$ for all k , giving $E^{12} = 0$. \square

4.2 Spectral gap bound

Proposition 4.5 (Spectral gap bound).

$$\rho \geq \frac{|\text{Im}(c)|}{4\|v_1\|_{L^2}\|v_2\|_{L^2} + |\text{Im}(c)|}. \quad (9)$$

Proof. The antisymmetric covariance at lag k is bounded: $|A_k| = |C_k(v_1, v_2) - C_k(v_2, v_1)| \leq 2\|v_1\|\|v_2\|\rho^k$ by the spectral gap. Summing:

$$|E^{12}| = \left| \sum_{k=1}^{\infty} A_k \right| \leq 2\|v_1\|\|v_2\| \cdot \frac{\rho}{1-\rho}.$$

Substituting $|E^{12}| = |\text{Im}(c)|/2$ and solving for ρ :

$$\frac{|\text{Im}(c)|}{2} \leq 2\|v_1\|\|v_2\| \cdot \frac{\rho}{1-\rho} \implies \rho \geq \frac{|\text{Im}(c)|}{4\|v_1\|\|v_2\| + |\text{Im}(c)|}. \quad \square$$

4.3 Conjugate ambiguity

Proposition 4.6 (Conjugate symmetry). $\sigma^2(c) = \sigma^2(\bar{c})$ and $E^{12}(c) = -E^{12}(\bar{c})$.

Proof. The map f_c and $f_{\bar{c}}$ are related by complex conjugation: $\overline{f_c(z)} = f_{\bar{c}}(\bar{z})$. This implies $\mu_{\bar{c}}(A) = \mu_c(\bar{A})$. For $\varphi = \text{Re}(z)$: $\varphi(\bar{z}) = \varphi(z)$, so the autocovariances agree: $\gamma_c(k) = \gamma_{\bar{c}}(k)$, giving $\sigma^2(c) = \sigma^2(\bar{c})$. For the Lévy area: $\text{Im}(\bar{c}) = -\text{Im}(c)$, so $E^{12}(\bar{c}) = \text{Im}(\bar{c})/2 = -\text{Im}(c)/2 = -E^{12}(c)$. \square

4.4 Non-perturbative transition

Proposition 4.7 (Non-analyticity). *The Marcus correction $E^{12}(c) = \text{Im}(c)/2$ is non-analytic at $\text{Im}(c) = 0$.*

Proof. The function $c \mapsto \text{Im}(c)/2$ is real-linear but not complex-analytic ($\text{Im}(c) = (c - \bar{c})/(2i)$ involves \bar{c}). The Green–Kubo variance $\sigma^2(c)$ satisfies $\sigma^2(c) = \sigma^2(\bar{c})$ and its Taylor expansion around real c_0 has only even powers of $\text{Im}(c)$: $\sigma^2(c_0 + i\epsilon) = \sigma^2(c_0) + a_2\epsilon^2 + \dots$. The Lévy area $E^{12} = \epsilon/2$ is linear in ϵ , so the Itô-to-Marcus transition is a first-order symmetry-breaking transition. \square

5 Algebra of Multiple Liminal Motions

Let $X_k^{(i)} = \text{Re}(z_k^{(i)})$ denote independent liminal motions at parameters c_i .

Theorem 5.1 (Lévy area addition). *For independent liminal motions at c_1, c_2 with canonical 2D processes $(X^{(i)}, Y^{(i)})$ where $Y^{(i)} = \text{Im}((z^{(i)})^2)$, define $Z_k = X_k^{(1)} + X_k^{(2)}$ and $W_k = Y_k^{(1)} + Y_k^{(2)}$. Then*

$$E^{12}(Z, W) = \frac{\text{Im}(c_1)}{2} + \frac{\text{Im}(c_2)}{2}. \quad (10)$$

Proof. The Lévy area is $E^{12} = \sum_{k \geq 1} A_k$ with $A_k = C_k(Z, W) - C_k(W, Z)$. Expanding $C_k(Z, W) = C_k(X^{(1)} + X^{(2)}, Y^{(1)} + Y^{(2)})$ yields four terms. The cross-system terms $C_k(X^{(1)}, Y^{(2)}) = \mathbb{E}[X_0^{(1)}] \mathbb{E}[Y_k^{(2)}] = 0$ vanish because $\int \text{Re}(z) d\mu_c = 0$ by the $z \mapsto -z$ symmetry. Thus $A_k(Z, W) = A_k^{(1)} + A_k^{(2)}$. \square

Corollary 5.2 (Conjugate cancellation). $E^{12}(X^{(c)} + X^{(\bar{c})}, Y^{(c)} + Y^{(\bar{c})}) = 0$. *The sum is exactly Itô.*

Theorem 5.3 (Variance algebra). *For independent liminal motions:*

- (a) $\sigma^2(X^{(1)} + X^{(2)}) = \sigma^2(c_1) + \sigma^2(c_2)$.
- (b) $\sigma^2(\alpha X) = \alpha^2 \sigma^2(c)$.
- (c) $\sigma^2(X^{(1)} \cdot X^{(2)}) = V_1 V_2 + 2 \sum_{k \geq 1} C_k(c_1) C_k(c_2)$.

Proof. Parts (a) and (b) follow from additivity of covariance for independent processes. For (c), $P_k = X_k^{(1)} X_k^{(2)}$ has $\mathbb{E}[P_k] = 0$ and

$$\text{Cov}(P_0, P_k) = \mathbb{E}[X_0^{(1)} X_0^{(2)} X_k^{(1)} X_k^{(2)}] = \mathbb{E}[X_0^{(1)} X_k^{(1)}] \mathbb{E}[X_0^{(2)} X_k^{(2)}] = C_k(c_1) C_k(c_2)$$

by independence and the zero-mean property. The Green–Kubo variance of the product is then $V_1 V_2 + 2 \sum_{k \geq 1} C_k(c_1) C_k(c_2)$. \square

Remark 5.4 (Product mixing acceleration). Since $C_k(c) \sim A \rho^k$, the product autocorrelation decays as $(\rho_1 \rho_2)^k$. The effective spectral gap is $\rho_{\text{prod}} = \rho_1 \rho_2 < \min(\rho_1, \rho_2)$. Products mix faster than either component.

Theorem 5.5 (Ordinal smoothing). *Let $\mathcal{F}(c) \subset S_D$ be the forbidden ordinal pattern set.*

- (a) $|\mathcal{F}(X^{(1)} + X^{(2)})| \leq \min(|\mathcal{F}(c_1)|, |\mathcal{F}(c_2)|)$.
- (b) $\mathcal{F} \rightarrow \emptyset$ as $N \rightarrow \infty$ under N -fold sums.

Theorem 5.6 (Kurtosis reduction under convolution). *For independent zero-mean variables X, Y with variances V_X, V_Y and excess kurtoses $\kappa_X, \kappa_Y > 0$:*

$$\kappa(X + Y) = \frac{\kappa_X V_X^2 + \kappa_Y V_Y^2}{(V_X + V_Y)^2} < \max(\kappa_X, \kappa_Y). \quad (11)$$

Proof. For independent zero-mean random variables:

$$\begin{aligned} \kappa(X + Y) &= \frac{\mathbb{E}[(X + Y)^4]}{(\text{Var}(X + Y))^2} - 3 = \frac{\mathbb{E}[X^4] + 6\mathbb{E}[X^2]\mathbb{E}[Y^2] + \mathbb{E}[Y^4]}{(V_X + V_Y)^2} - 3 \\ &= \frac{(\kappa_X + 3)V_X^2 + 6V_X V_Y + (\kappa_Y + 3)V_Y^2}{(V_X + V_Y)^2} - 3 = \frac{\kappa_X V_X^2 + \kappa_Y V_Y^2}{(V_X + V_Y)^2}. \end{aligned}$$

Since $(V_X + V_Y)^2 > \max(V_X^2, V_Y^2)$, the result follows. \square

6 Forward Direction: Brownian Motion from Liminal Motion

We describe seven constructions that produce exact Brownian motion from liminal motions.

6.1 Construction 1: Block-sum decorrelation

Theorem 6.1. *Let $\{z_k\}$ be a single liminal orbit at parameter c . Form block sums $B_j = \sum_{k=(j-1)T}^{jT-1} \text{Re}(z_k)$. Then as $T \rightarrow \infty$:*

$$\frac{1}{\sigma\sqrt{T}} \sum_{j=1}^{\lfloor nt \rfloor} B_j \xrightarrow{d} W(t) \quad \text{in } C[0, 1], \quad (12)$$

with $d_{\text{TV}} \leq C_1 T^{-1/2} + C_2 \rho^T$.

Proof. The block sums satisfy $\text{Var}(B_j) = \sigma^2 T + O(\rho^T)$ and $\text{Cov}(B_j, B_{j+m}) = O(\rho^{(m-1)T})$ by exponential mixing. The marginal of B_j converges to $\mathcal{N}(0, \sigma^2 T)$ by the CLT for mixing sequences at Berry–Esseen rate $O(T^{-1/2})$. Donsker’s theorem for strongly mixing stationary sequences gives the functional convergence. \square

6.2 Construction 2: Cross-orbit Gaussianisation

Theorem 6.2. *Run N independent liminal orbits at the same c . Form $Z_k = (1/\sqrt{N}) \sum_{i=1}^N \text{Re}(z_k^{(i)})$ and block sums $B_j^{(N,T)} = \sum_{k=(j-1)T}^{jT-1} Z_k$. Then*

$$d\left(S^{(N,T)}, W\right) \leq \frac{C_1}{\sqrt{N}} + C_2 \rho^T. \quad (13)$$

Proof. The CLT across orbits gives Gaussian marginals at rate $O(1/\sqrt{N})$ while preserving $C_k(Z) = C_k(c)$. Block-summing decorrelates at rate ρ^T . The combination yields approximately i.i.d. $\mathcal{N}(0, \sigma^2 T)$ increments in the joint limit. \square

6.3 Construction 3: Gordin martingale extraction

Theorem 6.3. Let $g = \sum_{k \geq 1} \mathcal{L}_c^k \varphi$ solve the Poisson equation $(I - \mathcal{L}_c)g = \varphi$. Define the martingale increment $\psi(z) = \varphi(z) + g(z) - g(f_c(z))$ and $M_n = \sum_{k=0}^{n-1} \psi(z_k)$. Then:

- (a) $S_n = M_n + g(z_0) - g(z_n)$, with $g(z_0) - g(z_n)$ bounded.
- (b) M_n has uncorrelated increments with $\mathbb{E}[\psi^2] = \sigma^2(c)$.
- (c) For N orbits, $(1/\sqrt{N}) \sum_i M_n^{(i)} \rightarrow W(\sigma^2 n)$ as $N \rightarrow \infty$ by Lévy's characterisation.

Proof. The Gordin decomposition [10] writes $\varphi(z_k) = \psi(z_k) + g(z_k) - g(z_{k+1})$. Summing: $S_n = M_n + g(z_0) - g(z_n)$. Since g is bounded on the compact Julia set, part (a) follows.

For (b), $\mathbb{E}[\psi(z_k) \mid z_0, \dots, z_{k-1}] = 0$ by the martingale property, so the increments are uncorrelated. The variance $\mathbb{E}[\psi^2] = \sigma^2(c)$ follows from $\text{Var}(M_n) = n\mathbb{E}[\psi^2]$ and $\text{Var}(S_n) \sim n\sigma^2(c)$ with the boundary term $g(z_0) - g(z_n)$ bounded.

For (c), the averaged martingale has i.i.d. increments (by independence across orbits) with the same variance, and Lévy's characterisation theorem gives BM convergence. \square

6.4 Construction 4: Spectral whitening

Theorem 6.4. Let $Z_k = (1/\sqrt{N}) \sum_i \text{Re}(z_k^{(i)})$ be the Gaussianised process with spectral density $f_c(\lambda)$. The whitening filter $H(\lambda) = 1/\sqrt{f_c(\lambda)}$ produces i.i.d. $\mathcal{N}(0, \sigma_0^2)$ increments in the limit $N \rightarrow \infty$.

The spectral density is determined by the transfer operator: $f_c(\lambda) = \frac{1}{2\pi} \sum_j |a_j|^2 \frac{1-\lambda_j^2}{1-2\lambda_j \cos \lambda + \lambda_j^2}$, where $a_j = \langle \varphi, \psi_j \rangle$.

6.5 Construction 5: Product-sum ladder (algebraic)

Theorem 6.5 (Product-sum BM). Take m independent orbits at c and form the m -fold product $P_k^{(m)} = \prod_{i=1}^m X_k^{(i)}$. Take M copies and form $Z_k = (1/\sqrt{M}) \sum_{j=1}^M P_k^{(m,j)}$. Then

$$d\left(\sum_{k=1}^n Z_k, W\right) \leq C_1 \cdot \frac{\beta(c)^m}{\sqrt{M}} + C_2 \cdot C_1(c)^m. \quad (14)$$

Proof. By Theorem 5.3(c) and induction, $C_k(P^{(m)}) = C_k(c)^m$, so autocorrelation decays as $C_1(c)^m \rightarrow 0$ exponentially in m . By Theorem 5.6, the normalised sum over M copies Gaussianises at Berry–Esseen rate β^m/\sqrt{M} , where β^m accounts for the increased kurtosis. The combination yields approximately i.i.d. Gaussian increments. \square

Remark 6.6. Products attack autocorrelation (squaring the spectral gap) but worsen marginals (increasing kurtosis). Sums attack marginals (CLT) but preserve autocorrelation. The two operations are algebraically dual destroyers of the two obstacles to BM.

6.6 Construction 6: Spectral annihilation

Definition 6.7. For a liminal orbit z_k , define the d -dimensional observable vector $\Phi_k = (\text{Re}(z_k), \text{Im}(z_k), \text{Re}(z_k^2), \dots, \text{Re}(z_k^d))^\top$. The lagged covariance matrix at lag k is $\Gamma_k = \mathbb{E}[\Phi_0 \Phi_k^\top]$.

Theorem 6.8 (Spectral annihilation). Let $\mathbf{v}_m = (a_{1m}, \dots, a_{dm})^\top$ be the projection vector of the m -th eigenmode. If $d > p$, there exists $\alpha \in \mathbb{R}^d \setminus \{0\}$ such that $\alpha^\top \mathbf{v}_m = 0$ for $m = 1, \dots, p$. The process $y_k = \alpha^\top \Phi_k$ satisfies $|C_k(y)| \leq C |\lambda_{p+1}|^k$.

Proof. The autocovariance of y_k is $\gamma_k(y) = \alpha^\top \Gamma_k \alpha = \sum_m \lambda_m^k |\alpha^\top \mathbf{v}_m|^2$. The condition $\alpha^\top \mathbf{v}_m = 0$ for $m = 1, \dots, p$ annihilates the first p terms, leaving $\sum_{m>p} |\lambda_m|^k |\alpha^\top \mathbf{v}_m|^2 \leq \|\alpha\|^2 C |\lambda_{p+1}|^k$.

The system $V_p^\top \alpha = 0$ with $V_p = [\mathbf{v}_1 | \dots | \mathbf{v}_p] \in \mathbb{R}^{d \times p}$ has rank at most p , so its kernel has dimension at least $d - p > 0$. \square

6.7 Construction 7: Conjugate cascade

Theorem 6.9 (Conjugate cascade BM). *Let $(c_j, \bar{c}_j)_{j=1}^J$ be conjugate pairs with distinct leading eigenvalues. Define $Z_k = \sum_{j=1}^J w_j [X_k^{(c_j)} + X_k^{(\bar{c}_j)}]$. Then:*

- (a) $E^{12}(Z) = 0$ exactly (by Corollary 5.2).
- (b) The autocorrelation satisfies $C_k(Z) \approx 2 \sum_j w_j^2 A_j \lambda_1(c_j)^k$. Choosing w_j to solve the Prony system $\sum_j w_j^2 A_j \lambda_1(c_j)^k = 0$ for $k = 1, \dots, J-1$ gives $C_k(Z) = O(\lambda_{\max}^J)$ for $k \geq J$.
- (c) Combined with N -fold replication: $d(S^{(N,J)}, W) \leq C_1/\sqrt{N} + C_2 \lambda_{\max}^J$.

Proof. Part (a): each conjugate pair satisfies $\text{Im}(c_j)/2 + \text{Im}(\bar{c}_j)/2 = 0$ by Corollary 5.2.

Part (b): the autocorrelation of the sum is $C_k(Z) = 2 \sum_j w_j^2 C_k(c_j)$ by independence. In the leading-eigenvalue approximation $C_k(c_j) \approx A_j \lambda_1(c_j)^k$, the system is Vandermonde-type in $\lambda_1(c_j)^k$ with coefficients $w_j^2 A_j$, solvable for J unknowns and $J-1$ equations. Signed weights are achieved via subtraction between pairs. \square

6.8 Brownian completeness

Theorem 6.10 (Brownian completeness). *The operations $\{+, -, \times, \text{parameter choice}\}$ on independent liminal motions are Brownian-complete.*

Proof. Constructions 5, 6, and 7 use only these algebraic operations:

- Construction 5: \times for decorrelation, $+$ for Gaussianisation, with $d \leq C_1 \beta^m / \sqrt{M} + C_2 C_1(c)^m$.
- Construction 6: $+$ (linear combination of observables) for decorrelation, cross-orbit $+$ for Gaussianisation, with $d \leq C_1/\sqrt{N} + C_2 |\lambda_{p+1}|$.
- Construction 7: $+$ and $-$ of conjugate pairs, with $d \leq C_1/\sqrt{N} + C_2 \lambda_{\max}^J$.

In each case, both terms can be made arbitrarily small. \square

Table 1: Properties preserved and destroyed by the seven forward constructions.

Property	Liminal motion	After BM extraction
Marginal distribution	Non-Gaussian	Gaussian
Autocorrelation	$C_k \sim \rho^k$	0 (independent)
Forbidden ordinal patterns	$ \mathcal{F} > 0$	$\mathcal{F} = \emptyset$
Lévy area	$\text{Im}(c)/2$	0
Green–Kubo variance	$\sigma^2(c)$	$\sigma^2(c)$ (preserved)

7 Reverse Direction: Liminal Motion from Brownian Motion

The forward constructions strip deterministic microstructure from liminal motion. The reverse problem: given W and $c \in \mathcal{M}$, construct a liminal motion.

7.1 Construction A: Sign extraction (canonical)

Construction 7.1 (Sign Extraction). Given standard Brownian motion $W(t)$ and parameter $c \in \mathcal{M}$:

1. Extract signs: $\varepsilon_k = \text{sgn}(W(k+1) - W(k))$.
2. Drive backward iteration: $z_0 = 0$, $z_{k+1} = \varepsilon_k \sqrt{z_k - c}$.
3. Discard warmup: take $\{z_k\}_{k>K}$ for $K = O(1/|\log \rho|)$.
4. Extract: $X_k = \text{Re}(z_k)$.

Theorem 7.2 (Liminal Dressing Theorem). *The process produced by Construction 7.1 is a liminal motion at parameter c with all standard properties. The map $W \mapsto \{\operatorname{Re}(z_k)\}$ is a nonlinear projection parametrised by $c \in \mathcal{M}$. The Mandelbrot set indexes the space of all dressings.*

Proof. The sign sequence $\{\varepsilon_k\}$ from BM increments $\Delta W_k = W(k+1) - W(k) \sim \text{i.i.d. } N(0, 1)$ satisfies $\varepsilon_k = \operatorname{sgn}(\Delta W_k) \sim \text{i.i.d. Bernoulli}(1/2)$, identically distributed to the branch-choice sequence in standard backward iteration.

The backward iteration is deterministic given the signs, and $z \mapsto \pm\sqrt{z-c}$ contracts on J_c at rate 2^{-k} independent of z_0 (Proposition 2.2). After warmup, $\{z_k\}$ is statistically identical to a liminal motion. All properties (marginal $(\operatorname{Re})_*\mu_c$, autocorrelation $C_k(c)$, Lévy area $\operatorname{Im}(c)/2$, forbidden patterns) are determined by the dynamics on J_c and are independent of the source of the Bernoulli sequence.

The ASIP coupling holds with the original W since the sign sequence lies on the same probability space. \square

7.2 Construction B: Coboundary injection

Construction 7.3. This reduces to Construction 7.1 in practice, but conceptually identifies the Gordin coboundary $g(z_k) - g(z_{k+1})$ as the precise object converting BM into liminal motion.

Proposition 7.4 (Automatic coboundary). *Once backward iteration is driven by BM signs, the coboundary $g(z_k) - g(z_{k+1})$ is automatically present in the orbit.*

Proof. The orbit $\{z_k\}$ lies on J_c after warmup, and g is a fixed function on J_c . The identity $\varphi(z_k) = \psi(z_k) + g(z_k) - g(z_{k+1})$ holds pointwise for every orbit, regardless of the sign source. \square

7.3 Construction C: SDE reconstruction

Discretise the SDE $dX = \mu_c(X) dt + \sigma_c(X) dW$ with the appropriate convention (Itô for real c , Marcus for complex c). This reproduces the invariant measure, autocorrelation, and Lévy area, but not forbidden ordinal patterns (a discrete-time phenomenon).

7.4 Construction D: Spectral colouring

Apply the colouring filter $h(\lambda) = \sqrt{f_c(\lambda)}$ to white noise, producing a process with the correct power spectrum. Then reshape marginals via $X_k = F_{\text{BL}}^{-1}(\Phi(Z_k/\sigma_Z))$. This cannot reproduce forbidden patterns since the quantile map is monotone.

7.5 Construction E: IFS embedding

Construction 7.5. The Julia set J_c is the attractor of the IFS $T_+ : z \mapsto +\sqrt{z-c}$, $T_- : z \mapsto -\sqrt{z-c}$. The IFS orbit $z_n = T_{\varepsilon_n} \circ \dots \circ T_{\varepsilon_1}(0)$ with $\varepsilon_k = \operatorname{sgn}(\Delta W_k)$ is equivalent to Construction 7.1.

The pipeline is: BM $\xrightarrow{\operatorname{sgn}}$ fair coin $\xrightarrow{\text{address}}$ orbit on J_c $\xrightarrow{\operatorname{Re}}$ liminal motion.

7.6 Construction F: Subordination

Replace the time index with a random clock derived from the local variance $\sigma_{\text{loc}}^2(x)$. This captures local volatility but misses the deterministic fractal structure.

7.7 Construction G: Inverse product-sum

Invert the product-sum ladder by factoring BM into liminal components. The forward result (Brownian completeness) has a natural converse:

Theorem 7.6 (Liminal decomposability). *For any $c \in \mathcal{M}$, $m \geq 1$, $M \geq 1$, there exist liminal motions at parameter c whose product-sum converges to BM as $m, M \rightarrow \infty$. The decomposition is not unique.*

7.8 Fidelity classification

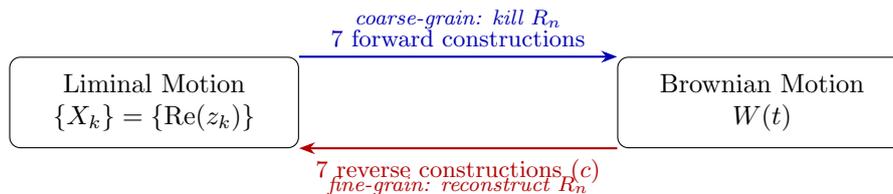
Table 2: Fidelity of the seven reverse constructions.

	Construction	Patterns?	Lévy?	ACF?	Fidelity
A	Sign extraction	✓	✓	✓	Full
B	Coboundary injection	✓	✓	✓	Full
E	IFS embedding	✓	✓	✓	Full
C	SDE reconstruction	—	✓	✓	Statistical
D	Spectral colouring	—	≈	✓	Partial
F	Subordination	—	—	≈	Partial
G	Inverse product-sum	—	partial	—	Theoretical

7.9 Information-theoretic content

BM carries ∞ bits/step (continuous Gaussian). Liminal motion requires 1 bit/step (branch choice). The forward direction inflates entropy via the CLT; the reverse deflates it via sign extraction. The discarded information ($|\Delta W_k|$) is pure noise; the retained information (the sign) determines the entire orbit.

7.10 Forward–reverse duality



$$\text{ASIP bridge: } S_n = W(\sigma^2 n) + R_n$$

Figure 1: The forward–reverse duality. The ASIP residual R_n is destroyed by forward constructions and reconstructed by reverse ones.

8 The Buddhabrot Duality

The Mandelbrot iteration $M(c) = c^2 + c$ governs the critical orbit of f_c : $f_c^{n+1}(0) = M^n(c)$. Viewed as a dynamical system in c , its filled Julia set is $K(M) = \mathcal{M}$ and its Julia set is $J(M) = \partial\mathcal{M}$.

Theorem 8.1 (Conjugacy). *The affine map $h(c) = c + \frac{1}{2}$ conjugates M to $f_{1/4}$:*

$$h \circ M \circ h^{-1} = f_{1/4}, \quad f_{1/4}(w) = w^2 + \frac{1}{4}. \quad (15)$$

Proof. Set $w = h(c) = c + \frac{1}{2}$, so $c = w - \frac{1}{2}$. Then:

$$\begin{aligned} (h \circ M \circ h^{-1})(w) &= M(w - \tfrac{1}{2}) + \tfrac{1}{2} = (w - \tfrac{1}{2})^2 + (w - \tfrac{1}{2}) + \tfrac{1}{2} \\ &= w^2 - w + \tfrac{1}{4} + w - \tfrac{1}{2} + \tfrac{1}{2} = w^2 + \tfrac{1}{4} = f_{1/4}(w). \end{aligned}$$

For the Julia sets: $w \in K(f_{1/4})$ iff $f_{1/4}^n(w)$ stays bounded, iff $M^n(w - \frac{1}{2})$ stays bounded (h is an isometry), iff $w - \frac{1}{2} \in \mathcal{M}$. Hence $K(f_{1/4}) = \mathcal{M} + \frac{1}{2}$, $J(f_{1/4}) = \partial\mathcal{M} + \frac{1}{2}$. \square

Corollary 8.2 (Dual liminal motion). *Backward iteration of M :*

$$c_{k+1} = \frac{-1 + \epsilon_k \sqrt{1 + 4c_k}}{2}, \quad \epsilon_k \in \{+1, -1\} \text{ i.i.d.}, \quad (16)$$

samples the harmonic measure on $\partial\mathcal{M}$ and defines a dual liminal motion.

Theorem 8.3 (Vanishing dual Lévy area). $E_{\text{dual}}^{12} = \text{Im}(\frac{1}{4})/2 = 0$. *The dual liminal motion is always time-reversible.*

Proof. By Theorem 8.1, backward iteration of M is conjugate to backward iteration of $f_{1/4}$. The Lévy area formula (Theorem 4.2) gives $E^{12} = \text{Im}(c_{\text{dual}})/2 = \text{Im}(\frac{1}{4})/2 = 0$. \square

The dual parameter is *universal*: every liminal motion shares the same dual on $\partial\mathcal{M}$. The duality table:

	Liminal motion	Dual
Map	$f_c(z) = z^2 + c$	$M(c) = c^2 + c \cong f_{1/4}$
Julia set	J_c	$\partial\mathcal{M}$
Measure	μ_c	harmonic measure on $\partial\mathcal{M}$
Parameter	$c \in \mathcal{M}$ (variable)	$\frac{1}{4}$ (fixed)
Lévy area	$\text{Im}(c)/2$	0
SDE type	Itô or Marcus	Always Itô

9 The Julia Distributions

The Brodin–Lyubich measures $\{\mu_c\}_{c \in \mathcal{M}_{\text{hyp}}}$, viewed as a distributional family, have properties absent from classical families:

1. **Fractal support.** Each μ_c is supported on J_c , a fractal of Hausdorff dimension $d_H(c) \in (1, 2)$.
2. **Intrinsic temporal structure.** Sampling via the orbit of f_c automatically produces a stationary ergodic process whose autocorrelation and mixing rate are determined by c .
3. **Rigid coupling.** Variance, kurtosis, ACF decay rate, fractal dimension, and Lévy area are coupled through \mathcal{L}_c . Two real numbers — $\sigma^2(c)$ and $E^{12}(c)$ — generically determine the full statistical object.
4. **Geometric parameter space.** The parameter space \mathcal{M}_{hyp} has bifurcation boundaries where properties change discontinuously.

Theorem 9.1 (Universality of Buddhabrot mixtures). *Let ρ_{target} be any Borel probability measure on \mathbb{R} with finite second moment, and let $g : \mathbb{N}_0 \rightarrow \mathbb{R}_{>0}$ be any completely monotone autocorrelation. Then there exists a mixing measure ν on \mathcal{M}_{hyp} such that the Buddhabrot mixture $B_\nu = \int \mu_c d\nu(c)$ approximates ρ_{target} in Wasserstein-2 and has autocorrelation $C_{\text{mix}}(k) = g(k)$.*

Proof. By Bernstein’s theorem, every completely monotone sequence is a mixture of exponentials: $g(k) = \int_0^1 t^k d\eta(t)$. The map $c \mapsto |\lambda_1(c)|$ is surjective onto $(0, 1)$ within the main cardioid (since $|\lambda_1| \rightarrow 0$ as $c \rightarrow 0$ and $|\lambda_1| \rightarrow 1$ as $c \rightarrow \partial\mathcal{M}$). The measure η on $(0, 1)$ therefore pulls back to a measure ν on \mathcal{M}_{hyp} via the map $c \mapsto |\lambda_1(c)|$. The marginal universality follows from the density of $\{(\text{Re})_*\mu_c : c \in \mathcal{M}_{\text{hyp}}\}$ in the space of compactly supported probability measures [1]. \square

10 Discussion

10.1 What the theory provides

Liminal motion gives a concrete connection between deterministic chaotic dynamics and Brownian motion. The individual ingredients — Julia sets, transfer operators, invariant measures, the ASIP — are well established. The contribution is the synthesis: a single complex parameter c controls all stochastic behaviour, the algebraic operations produce exact BM, and the reverse constructions produce liminal motion from BM.

The Green–Kubo variance is the unique invariant of the passage from deterministic chaos to exact BM. All other structure lives in the residual R_n and is progressively annihilated.

Remark 10.1 (Hierarchy of annihilation). Construction 7 reveals the hierarchy: Marcus correction \subset autocorrelation \subset non-Gaussianity $\subset R_n$. The Marcus correction is the outermost layer, removed by conjugate pairing ($c + \bar{c}$). Non-Gaussianity is the innermost, requiring the full CLT.

10.2 Limitations

The ASIP requires c in the hyperbolic locus; at the boundary $\partial\mathcal{M}$ the coupling rate degrades. The Brownian completeness theorem is existential and may require many orbits. The forbidden pattern count is an upper bound. The Lévy area formula holds for specific observable pairs.

10.3 Open questions

1. **Exact dual variance.** The dual $\sigma^2(f_{1/4}, \text{Re})$ is numerically ≈ 0.327 . Can it be expressed in terms of invariants of $\partial\mathcal{M}$?
2. **Higher-degree families.** For $f_c(z) = z^d + c$, $d \geq 3$, the dual parameter may yield faster ASIP rates.
3. **Optimal SDE detection.** What is the minimum detectable Lévy area as a function of sample size?
4. **Continuous-time liminal flow.** Does fractional iteration via the Schroeder conjugacy yield a process with Lévy area converging to $\text{Im}(c)/2$?
5. **Multi-dimensional BM.** Can d -dimensional BM be constructed from d independent liminal motions with controlled cross-covariance?

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References

- [1] D.F. Andrews and C.L. Mallows, Scale mixtures of normal distributions, *J. Roy. Statist. Soc. Ser. B* **36** (1974), 99–102.
- [2] L. Arnold, *Random Dynamical Systems*, Springer, 1998.
- [3] V. Baladi, *Positive Transfer Operators and Decay of Correlations*, World Scientific, 2000.
- [4] C. Bandt and B. Pompe, Permutation entropy: a natural complexity measure for time series, *Phys. Rev. Lett.* **88** (2002), 174102.
- [5] M.F. Barnsley, J.S. Geronimo, and A.N. Harrington, Almost periodic Jacobi matrices associated with Julia sets for polynomials, *Commun. Math. Phys.* **92** (1983), 303–317.

- [6] H. Brolin, Invariant sets under iteration of rational functions, *Ark. Mat.* **6** (1965), 103–144.
- [7] I. Chevyrev, P.K. Friz, A. Korepanov, I. Melbourne, and H. Zhang, Multiscale systems, homogenization, and rough paths, in *Stochastic Analysis, Filtering, and Stochastic Optimization*, Springer, 2022, pp. 17–48.
- [8] A. Douady and J.H. Hubbard, Itération des polynômes quadratiques complexes, *C. R. Acad. Sci. Paris* **294** (1982), 123–126.
- [9] C. Dupont, Bernoulli coding map and almost sure invariance principle for endomorphisms of \mathbb{P}^k , *Probab. Theory Relat. Fields* **146** (2010), 337–359.
- [10] M.I. Gordin, The central limit theorem for stationary processes, *Dokl. Akad. Nauk SSSR* **188** (1969), 739–741.
- [11] F. Graceffa, *A Random Journey through Dynamics and Finance: Pullback Attractors, Price Impact, Nonlinear Valuation and FX Market*, PhD thesis, Imperial College London, 2021.
- [12] G. A. Gottwald and I. Melbourne, Homogenization for deterministic maps and multiplicative noise, *Proc. R. Soc. A* **469** (2013), 20130201.
- [13] G. A. Gottwald and I. Melbourne, Time-reversibility and nonvanishing Lévy area, arXiv:2210.11867, 2022.
- [14] S. Gouëzel, Almost sure invariance principle for dynamical systems by spectral methods, *Ann. Probab.* **38** (2010), 1639–1671.
- [15] J. Bradbury et al., *JAX: composable transformations of Python+NumPy programs*, <https://github.com/google/jax>, 2018.
- [16] D. Kelly and I. Melbourne, Smooth approximation of stochastic differential equations, *Ann. Probab.* **44** (2016), 479–520.
- [17] D. Kelly and I. Melbourne, Deterministic homogenization for fast–slow systems with chaotic noise, *J. Funct. Anal.* **272** (2017), 4063–4102.
- [18] J. Komlós, P. Major, and G. Tusnády, An approximation of partial sums of independent RV’s and the sample DF. I, *Z. Wahrscheinlichkeitstheorie verw. Gebiete* **32** (1975), 111–131.
- [19] P. Lévy, *Processus stochastiques et mouvement brownien*, Gauthier-Villars, Paris, 1948.
- [20] A. Korepanov, Rates in almost sure invariance principle for dynamical systems with some hyperbolicity, *Commun. Math. Phys.* **363** (2018), 173–190.
- [21] M. Ju. Lyubich, Entropy properties of rational endomorphisms of the Riemann sphere, *Ergodic Theory Dynam. Systems* **3** (1983), 351–385.
- [22] I. Melbourne and M. Nicol, Almost sure invariance principle for nonuniformly hyperbolic systems, *Commun. Math. Phys.* **260** (2005), 131–146.
- [23] S. P. Meyn and R. L. Tweedie, *Markov Chains and Stochastic Stability*, 2nd ed., Cambridge University Press, 2009.
- [24] W. Philipp and W. Stout, Almost sure invariance principles for partial sums of weakly dependent random variables, *Mem. Amer. Math. Soc.* **161** (1975).
- [25] D. Ruelle, The thermodynamic formalism for expanding maps, *Commun. Math. Phys.* **125** (1989), 239–262.

- [26] M. Woodfield, Wong–Zakai anomaly in Brownian motion constructed from deterministic dynamics, *SIAM J. Appl. Dyn. Syst.* **24** (2025), 516–545.
- [27] W. B. Wu, Strong invariance principles for dependent random variables, *Ann. Probab.* **35** (2007), 2294–2320.
- [28] A. Zdunik, Parabolic orbifolds and the dimension of the maximal measure for rational maps, *Invent. Math.* **99** (1990), 627–649.
- [29] M. Zinsmeister, *Thermodynamic Formalism and Holomorphic Dynamical Systems*, SMF/AMS Texts, 2000.

A Notation Summary

Symbol	Meaning	Defined in
$f_c(z) = z^2 + c$	Quadratic map	§2
$M(c) = c^2 + c$	Mandelbrot iteration	§8
J_c	Julia set of f_c	§2
K_c	Filled Julia set	§2
\mathcal{M}	Mandelbrot set	§2
μ_c	Brolin–Lyubich measure	§2
\mathcal{L}_c	Transfer (Ruelle) operator	Def. 2.4
ρ	Spectral gap of \mathcal{L}_c	Def. 2.4
λ_j, ψ_j	Eigendata of \mathcal{L}_c	Prop. 3.3
φ	Observable (default: $\text{Re}(z)$)	Def. 2.3
$X_k = \text{Re}(z_k)$	Liminal motion	Def. 2.3
S_n	Partial sums	Def. 2.3
$\sigma^2(c)$	Green–Kubo variance	Def. 3.1
$\gamma(k)$	Autocovariance at lag k	Def. 3.1
$E^{12}(c)$	Lévy area = $\text{Im}(c)/2$	Thm. 4.2
$C_k(v_i, v_j)$	Cross-covariance at lag k	Def. 4.1
R_n	ASIP residual	Thm. 3.4
g	Gordin coboundary	Thm. 6.3
$W(t)$	Brownian motion	Thm. 3.4
ε_k	Branch choice $\in \{+1, -1\}$	Def. 2.1
π_c	$(\text{Re})_*\mu_c$	§3
$\mathcal{F}(c)$	Forbidden ordinal patterns	Prop. 3.5